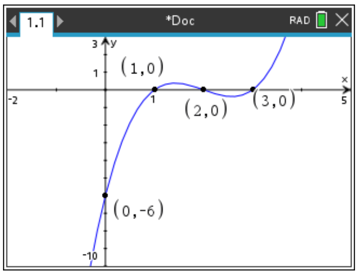
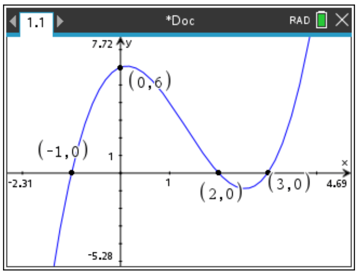
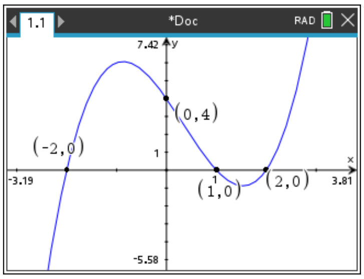
Investigation: Product of Polynomials – Possible Responses

Component 1

In this component you will consider graphs of cubic polynomials.

1. Consider the family of curves of the form $y = (x - a)(x - b)(x - c)$ where a, b, c are real numbers $(a, b, c \in \mathbb{R})$.

   a. i. By selecting your own values for a, b, c, where $a \neq b \neq c$, sketch 3 cubic graphs of the above form. Label your axial intercepts with coordinates. Also write the equation for each corresponding graph.

<table>
<tr>
<td>$a = 1, \quad b = 2, \quad c = 3$</td>
<td>$a = -1, \quad b = 2, \quad c = 3$</td>
<td>$a = -2, \quad b = 1, \quad c = 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y = (x - 1)(x - 2)(x - 3)$</td>
<td>$y = (x + 1)(x - 2)(x - 3)$</td>
<td>$y = (x + 2)(x - 1)(x - 2)$</td>
</tr>
</table>

   ii. Comment on any similarities/differences between your graphs.

Similarities:

- They all have 3 x-intercepts
- They all have 2 turning points
- They all have the same shape

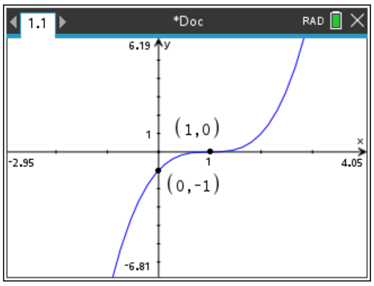
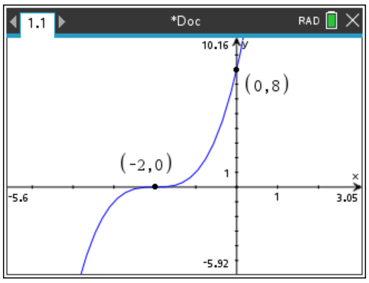
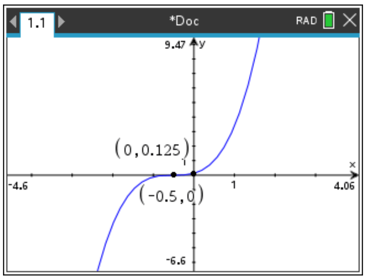
Differences:

- Some have positive y-intercepts while others have negative y-intercepts
- Some have all positive x-intercepts, some have all negative x-intercepts, and some have both positive and negative x-intercepts

   iii. Discuss how a, b, c affect the key features of the graph.

The values pf a, b, c correspond to the values of the x-intercepts

b. What happens to the shape and key features of the cubic graph if $a = b = c$? Investigate. Provide 3 examples to support your ideas/conjectures. Label your axial intercepts with coordinates. Also write the equation for each corresponding graph.
Comment on any similarities/differences. Try to generalise your observations.

<table>
<tr><td>$a = b = c = 1$</td><td>$a = b = c = -2$</td><td>$a = b = c = -\frac{1}{2}$</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>$y = (x-1)^3$</td><td>$y = (x+2)^3$</td><td>$y = \left(x + \frac{1}{2}\right)^3$</td></tr>
</table>

Similarities:

- All have the same shape
- All have just one x-intercept
- All have a stationary point of inflection at the x-intercept

Differences:

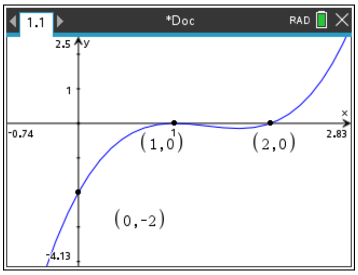
- Some have positive y-intercepts while others have negative y-intercepts
- Some have positive x-intercepts, some have negative x-intercepts

Generalisations:

- All graphs of the form $y = (x-a)^3$ will have a stationary point of inflection at the x-intercept
- The graph/stationary point of inflection/x-intercept will move to the right if $a > 0$ and will move to the left if $a < 0$
- The y-interept of $y = (x-a)^3$ will be positive if $a < 0$ and negative if $a > 0$

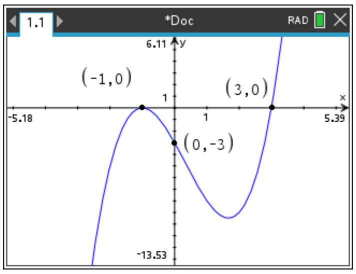
c. What happens to the shape and key features of the cubic graph if $a = b \neq c$? Investigate. Provide 3 examples to support your ideas/conjectures. Label your axial intercepts with coordinates. Also write the equation for each corresponding graph.
Comment on any similarities/differences. Try to generalise your observations.

$a = b = 1, \quad c = 2$



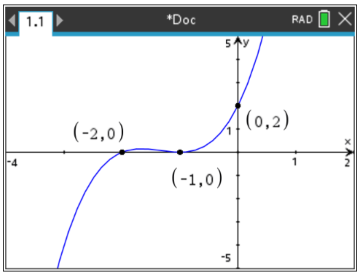
$y = (x-1)^2(x-2)$

$a = b = -1, \quad c = 3$



$y = (x+1)^2(x-3)$

$a = b = -1, \quad c = -2$



$y = (x+1)^2(x+2)$

Similarities:

- All have the same shape
- All have just two x-intercept
- Some students may use CAS technology to identify that all of these graphs will have a (non-stationary) point of inflection

Differences:

- Some have positive y-intercepts while others have negative y-intercepts
- Some have positive x-intercepts, some have negative x-intercepts

Generalisations:

- All graphs of the form $y = (x-a)^2(x-c)$ will have a turning point at $x = a$ (or at the point $(a, 0)$), and the graph will cut through the x-axis at $x = c$ (or the point $(c, 0)$).
- The graph will have a positive y-intercept if c is negative, and a negative y-intercept if c is positive

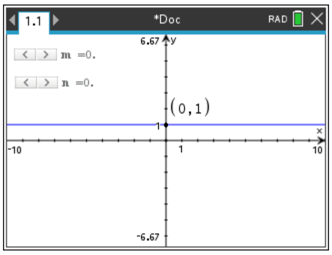
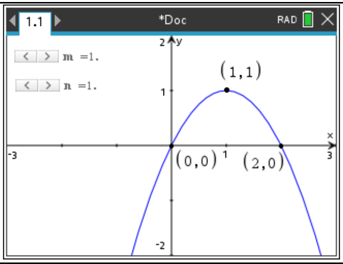
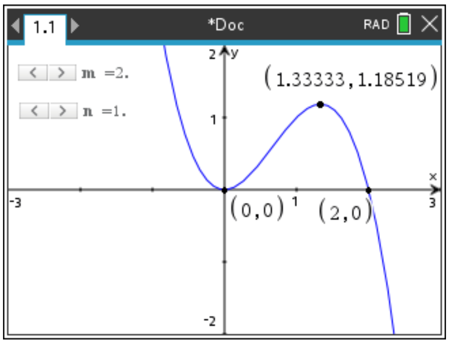
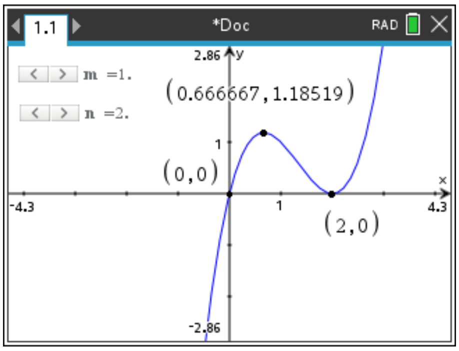
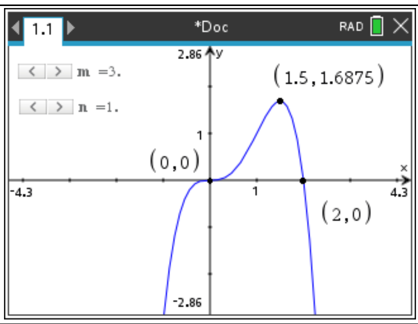
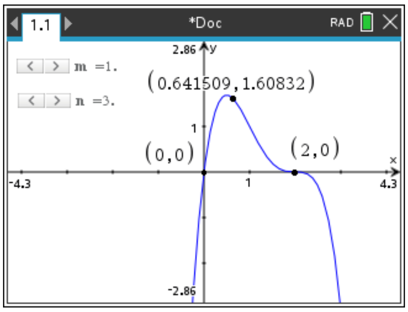
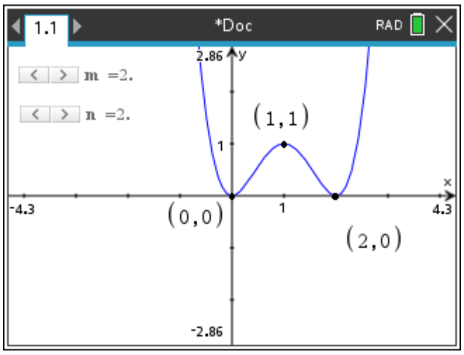
2. Consider the family of curves of the form $y = (x+1)(ax^2 + bx + c)$, where a, b, c are non-zero real numbers.
Investigate how a, b, c affect the shape and key features of the graph.
Provide examples to support your ideas/conjectures. Summarise/describe your observations, and try to generalise your observations.

- c corresponds to the y-intercepts
- All graphs will pass through the x-intercept $(1,0)$
- $\Delta = b^2 - 4ac$ will determine the number of intercepts of the graph
  - $\Delta < 0$ will give one x-intercept at $(1,0)$
  - $\Delta = 0$ will give two x-intercepts
  - $\Delta > 0$ will give three x-intercepts

Component 2

In this component you will consider graphs of the form $y = x^m(h - x)^n$, where h is non-zero real number and $m, n \in \{0, 1, 2, 3, 4\}$

1. Consider the case where $h = 2$.
   a. Sketch the following graphs, labelling all key features.

$m = 0, \quad n = 0 \quad y = x^0(2 - x)^0$	$m = 1, \quad n = 1 \quad y = x(2 - x)$
	
$m = 2, \quad n = 1 \quad y = x^2(2 - x)$	$m = 1, \quad n = 2 \quad y = x(2 - x)^2$
	
$m = 3, \quad n = 1 \quad y = x^3(2 - x)$	$m = 1, \quad n = 3 \quad y = x(2 - x)^3$
	
$m = 2, \quad n = 2 \quad y = x^2(2 - x)^2$	
	

b. Comment on any similarities/differences between your graphs.

Similarities:

- All graphs (except for the case of $m = n = 0$) pass through the points $(0,0)$ and $(2,0)$/have two x-intercepts
- The graphs of $m = 2$ & $n = 1$ and $m = 1$ & $n = 2$ (i.e. $m + n = 3$) have a cubic shape.
- The graphs of $m = 3$ & $n = 1$ and $m = 1$ & $n = 3$ have a point of inflection at one of the x-intercepts (at the intercept whose factor has degree 3).
- There is a turning point when at least one of the powers is greater than or equal to 2 (2 or more).
- There is a stationary point of inflection when one of the powers is 3

Differences:

- ~~Some have positive y-intercepts while others have negative y-intercepts~~
- ~~Some have positive x-intercepts, some have negative x-intercepts~~

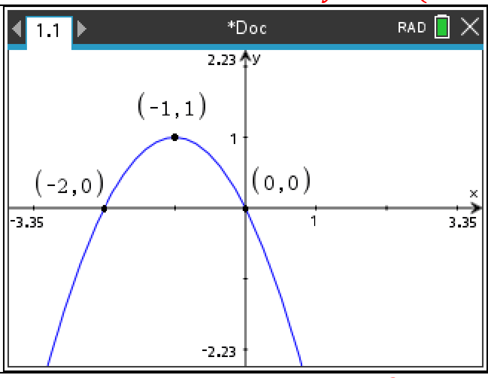
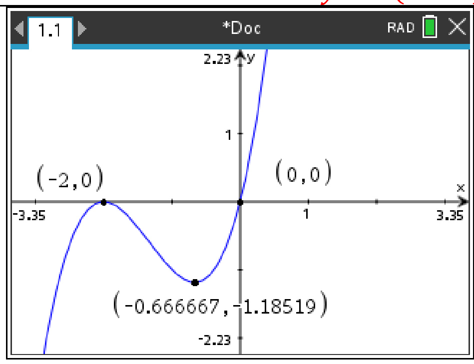
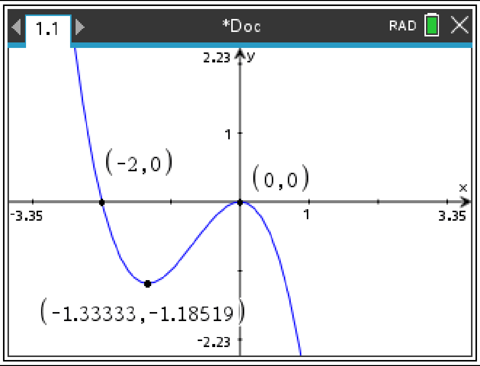
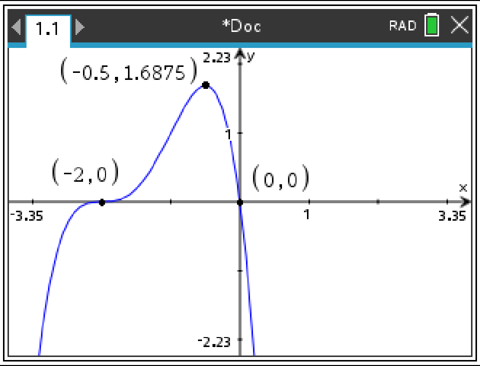
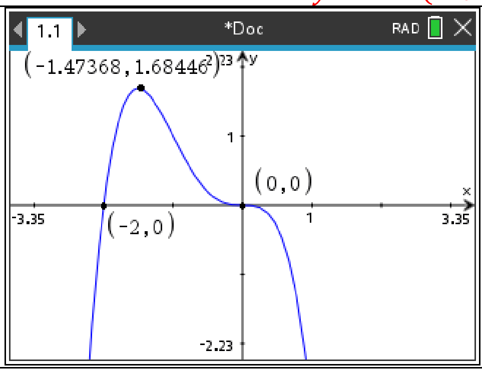
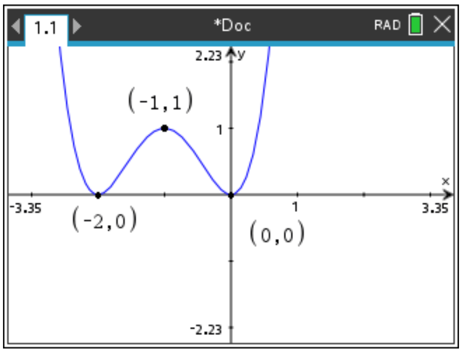
c. Discuss how m and n affect:
    - The behaviour/shape of the graph. Provide examples to support your ideas/conjectures (select a different value of h).
    - The number and nature of any turning points/point of inflection. Provide examples to support your ideas/conjectures (select a different value of h).

- m & n affect the behaviour of the graph at the x-intercepts
- When m or n is odd, the graph cuts through the x-intercept
    - If m or n is 3, there is also a stationary point of inflection at the intercept that has the power of 3
- When m or n is even, the graph will have a turning point at the x-interept that corresponds to that power (factor).

2. What would happen to the graph and key features if h was negative? Investigate.

Students could try a particular value of h (or values of h), compare with their earlier results and then notice any patterns emerging to generalise their results/observations.

Consider $h = -2$

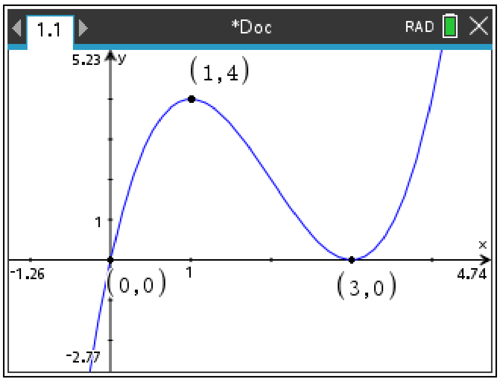
$m = 1, \quad n = 1$ $\quad y = x(-2 - x)$ $y = -x(x + 2)$	$m = 1, \quad n = 2$ $\quad y = x(-2 - x)^2$ $y = x(x + 2)^2$
	
$m = 2, \quad n = 1$ $\quad y = x^2(-2 - x)$ $y = -x^2(x + 2)$	$m = 1, \quad n = 3$ $\quad y = x(-2 - x)^3$ $y = -x(x + 2)^3$
	
$m = 3, \quad n = 1$ $\quad y = x^3(-2 - x)$ $y = -x^3(x + 2)$	$m = 2, \quad n = 2$ $\quad y = x^2(-2 - x)^2$ $y = x^2(x + 2)^2$
	

- For $h = -2$, the x-intercepts are now located at $(-2, 0)$ and $(0,0)$
- The x-intercepts are located at $(h, 0)$ and $(0,0)$
- Graphs where $m + n$ is even appear to be reflected in the y-axis, however this is not the case for graphs where $m + n$ is odd (Note: this could be extended into exploring odd and even functions)
- The number of turning points and points of inflection are the same as when h is positive, however their locations are different

3. Create an equation of a graph of the form $y = x^m(h-x)^n$ that satisfies the following conditions:

Graph 1: Has an x-intercept and turning point at $x = 3$

One possible answer: $y = x(3-x)^2$



Anything of the form $y = x^m(3-x)^n$ where $m \in \mathbb{N}$ and $n \in 2\mathbb{N}$ will work

Graph 2:

- Has at least 2 turning points
- Has a negative x-intercept

Possible answers could include:

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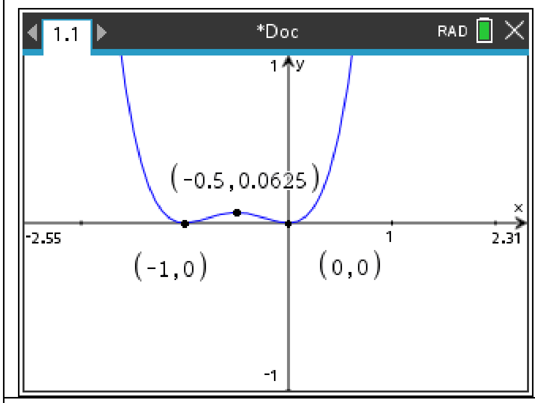
$y = x^2(-1-x)^2$



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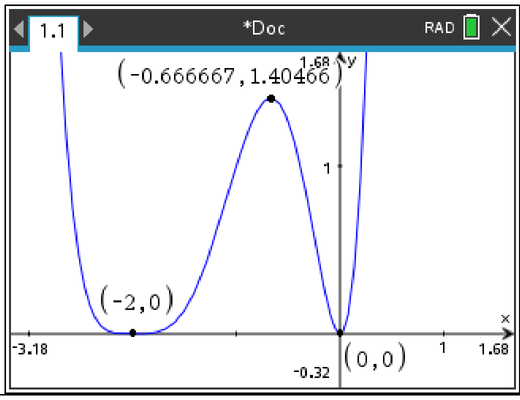
$y = x^2(-2-x)^4$



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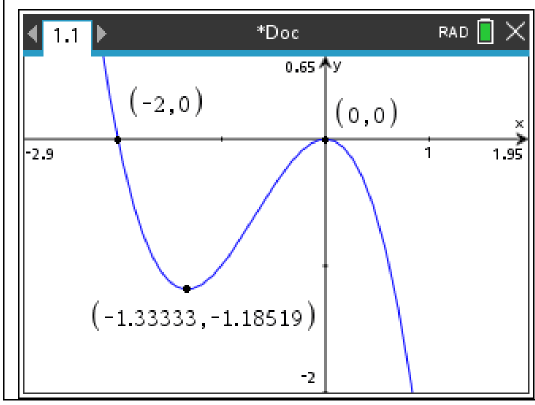
$y = x^2(-2-x)$



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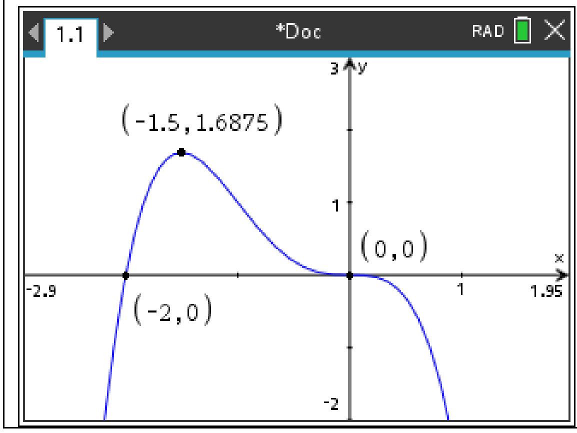
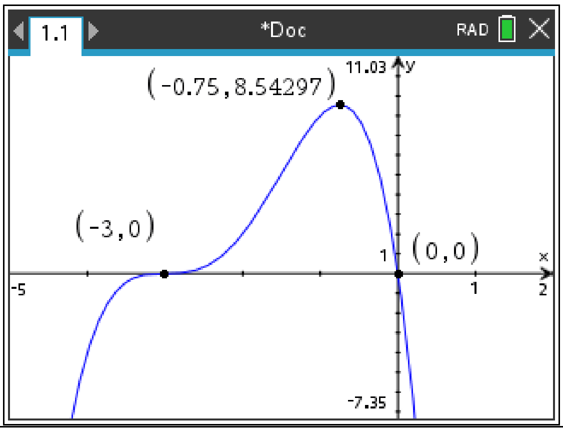
</table>

Graph 3:

- Has a stationary point of inflection
- Has a negative x-intercept

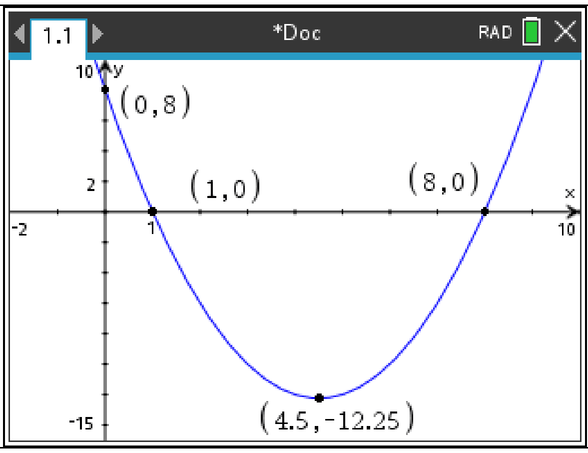
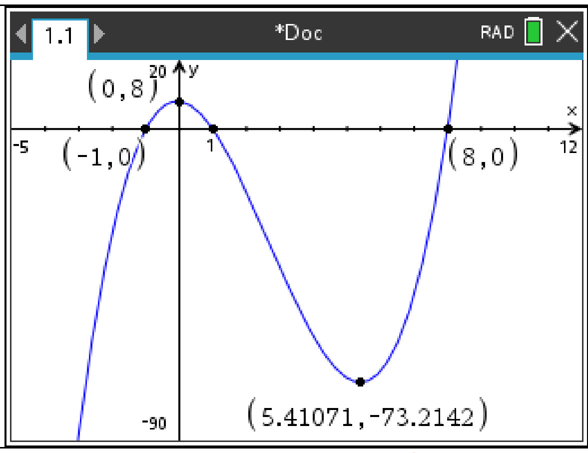
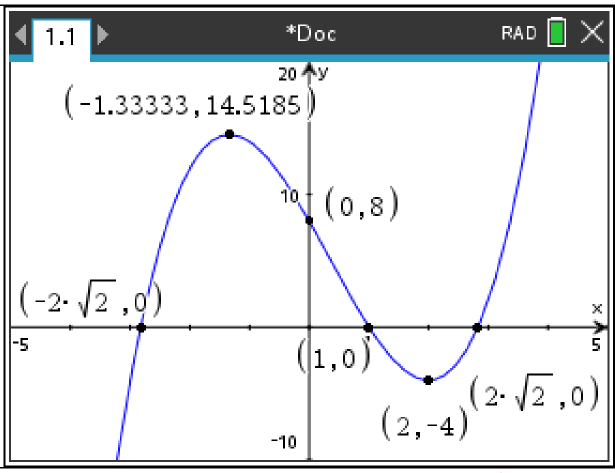
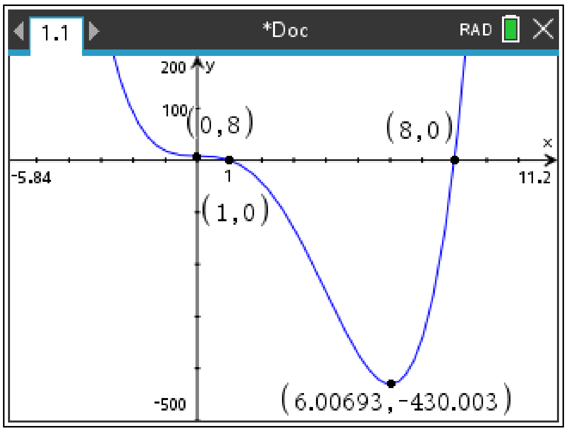
Possible answers could include:

$y = x^3(-2 - x)$	$y = x(-3 - x)^3$
	

Component 3

In this component you will consider graphs of the form $y = (x^m - a)(x^n - b)$,
where a, b are non-zero real numbers and $m, n \in \{1, 2, 3\}$.

a. Consider the family of curves of the form $y = (x^m - 1)(x^n - 8)$ where $2 \leq m + n \leq 4$.
Investigate how m and n affect:
- The location and number of axial intercepts
- The behaviour of the graph
- (The number of turning points/stationary points of inflection)

<table>
<tr><td>$m = 1, \quad n = 1 \qquad y = (x-1)(x-8)$</td><td>$m = 1, \quad n = 2 \qquad y = (x-1)(x+1)(x-8)$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$m = 2, \quad n = 1 \qquad y = (x-1)(x-\sqrt{8})(x+\sqrt{8})$</td><td>$m = 1, \quad n = 3 \qquad y = (x-1)(x^2+x+1)(x-8)$</td></tr>
<tr><td>
</td><td>
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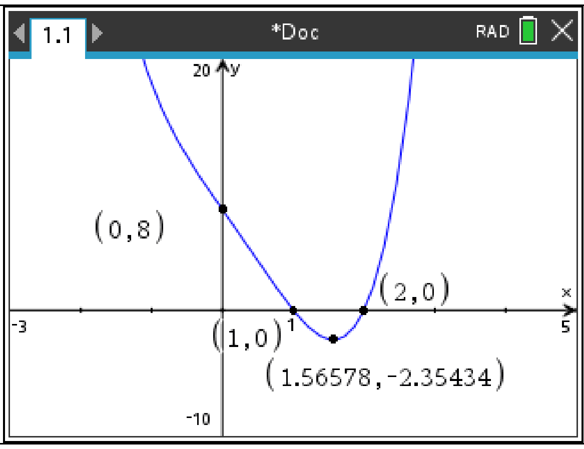
<tr><td>$m = 3, \quad n = 1 \qquad y = (x-1)(x-2)(x^2+2x+4)$</td><td></td></tr>
<tr><td>
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The location and number of axial intercepts

- In all cases, there is an x-intercept at $x = 1$ (at the point $(1, 0)$)
- In all cases, there is a y-intercept at $y = 8$ (at the point $(0, 8)$)
- In all cases, there two positive x-intercepts
- When $m + n = 2$ or 4 (or $m + n \in 2\mathbb{N}$) there are only 2 x-intercepts, located at $(1,0)$ and $\left(8^{\frac{1}{m}}, 0\right)$ or $\left(\sqrt[m]{8}, 0\right)$
- When $m + n = 3$ $(m + n \in 2\mathbb{N} + 1)$ there are three x-intercepts, since the factors are one that is linear and one that is a difference of perfect squares
- When $n > m$, the x-intercepts are at $(\pm 1, 0)$ and $\left(8^{\frac{1}{m}}, 0\right)$
- When $m > n$, the x-intercepts are at $(1,0)$ and $\left(\pm 8^{\frac{1}{m}}, 0\right)$

The behaviour of the graph

- When $m + n = 2$ or 4, the graph has a quadratic or quartic shape ($'U'$ shape)
- When $m + n = 3$, the graph has a cubic shape

The number of turning points/stationary points of inflection

- When $m + n = 2$, there is one turning point
- When $m + n = 3$, there are two turning points (and one non-stationary point of inflection)
- When $m + n = 4$, there is one turning point and one point of inflection

b. Consider the family of curves of the form $y = (x^m - 1)(x^n - a)$ where $2 \leq m + n \leq 4$ and a is a non-negative real number.
Investigate how a, m and n affect:

- The location and number of axial intercepts
- The behaviour of the graph
- (The number of turning points/stationary points of inflection)

End of Investigation